

Equilibrium Solution to the Lowest Unique Positive Integer Game

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We address the equilibrium concept of a reverse auction game so that no one can enhance the individual payoff by a unilateral change when all the others follow a certain strategy. In this approach the combinatorial possibilities to consider become very much involved even for a small number of players, which has hindered a precise analysis in previous works. We here present a systematic way to reach the solution for a general number of players, and show that this game is an example of conflict between the group and the individual interests.

Keywords: Lowest unique positive integer game; Nash equilibrium; projection operator; social dilemma

1. Introduction

Game theory deals with a situation where each player's payoff is dependent not only on her own behavior but also on other players'. Players will generally have conflicting interests with each other, but they may get better off by interacting with others in various ways [1]. For example, in the minority game [2], a player gets a point when her choice is minor among players. Thus, even if a choice happened to be successful at previous rounds, it is hard to remain minor with that choice as more and more players will also select it. Although it is not possible for all the players to win together, the average probability of winning can become enhanced if they behave in proper ways.

Recently, an extreme version of the minority game was proposed as a special case of the reverse auction, which is called the *lowest unique positive integer* (LUPI) game [3]. This game has $n > 2$ players, and each of them may choose one integer from 1 to n . A player wins a point by choosing the lowest unique number. That means, simply choosing the number 1 is not a good strategy since it is very likely to be chosen by other players, too. In Ref. [4], this LUPI game was analyzed in terms of the Nash equilibrium (NE): Let us imagine that every player participates

in a game with her own strategy, which is publicly known to the other players. If no player can improve her payoff by changing her strategy alone, the set of the players' strategies is called a NE. Even though this is an adequate framework to analyze a game, the actual enumeration of combinatorial possibilities becomes hard to manage as the number of players increases. Although Ref. [4] tried to find an expression for general n , many of the possible cases were missing in the probability calculations, even for $n = 4$. According to the suggested formula in Ref. [4], all the losing bids should be above the winning number or all should be below the winning number. Furthermore, if they are below the winning number, they should all be on the same number. However, there exist many more cases that would generate the same winning number. We henceforth present a way to take all the possibilities into account correctly for general n and analyze the results thereby obtained.

2. Equilibrium

Suppose that each player draws numbers from her own probability distribution, which we will denote as her strategy. As explained above, we will find a NE in this strategy space. Among several different NE's in the LUPI game, we will be concerned with one that can be prevalent among these n players, as studied in Ref. [4]. This approach actually corresponds to the stability concept in the evolutionary game theory [5]. Let us imagine that $n - 1$ people are using a certain strategy, $\vec{p} = (p_1, p_2, \dots, p_n)$, where p_i means the probability to choose number i . The idea is to choose \vec{p} so that the chance of winning for the n th player becomes independent of her strategy. We consider all the possible cases that those $n - 1$ people can make. These can be symbolically represented by

$$Z_0 = \left(\sum_{i=1}^n p_i \right)^{n-1}, \quad (1)$$

from which all the combinations can be obtained as separate terms. For example, in a three-players game we get $Z_0 = (p_1 + p_2 + p_3)^2 = p_1^2 + p_2^2 + p_3^2 + 2p_1p_2 + 2p_1p_3 + 2p_2p_3$, where the first three terms on the right-hand side mean that players 1 and 2 chose the same number 1, 2 or 3, respectively. The other three terms correspond to the cases when players 1 or 2 have chosen two different numbers. Together, these terms represent all the possible outcomes of the game (as seen from the third player) since they add up to the probability 1 due to the normalization condition $\sum_i^n p_i = 1$.

Now, starting from Eq. (1), we extract all the cases where there is a winner, among the $n - 1$ players, at number 1. These are characterized by all the terms which contain only one p_1 . In other words, if we write Eq. (1) as a polynomial in p_1 :

$$Z_0 = A(p_2, \dots, p_n) + B(p_2, \dots, p_n)p_1 + C(p_2, \dots, p_n)p_1^2 + \dots,$$

those cases are expressed by the second term and one can easily find it by $B =$

$dZ_0/dp_1|_{p_1=0}$. In other words, we find the cases with no winner at number 1 as

$$\begin{aligned} Z_1 &= Z_0 - \frac{dZ_0}{dp_1} \Big|_{p_1=0} p_1 \\ &= \left(\sum_{i=1}^n p_i \right)^{n-1} - (n-1)p_1 \left(\sum_{i \neq 1}^n p_i \right)^{n-2}. \end{aligned}$$

If we are further to exclude the cases where there is a winner at number 2, we apply the same operation on Z_1 as follows:

$$\begin{aligned} Z_2 &= Z_1 - \frac{dZ_1}{dp_2} \Big|_{p_2=0} p_2 \\ &= \left(\sum_{i=1}^n p_i \right)^{n-1} - (n-1)p_1 \left(\sum_{i \neq 1}^n p_i \right)^{n-2} \\ &\quad - (n-1)p_2 \left(\sum_{i \neq 2}^n p_i \right)^{n-2} + (n-1)(n-2)p_1 p_2 \left(\sum_{i \neq 1,2}^n p_i \right)^{n-3}. \end{aligned}$$

Generalizing this idea, we can find cases where there is no winner up to an arbitrary number i by the following recursion relation:

$$Z_i = Z_{i-1} - \frac{dZ_{i-1}}{dp_i} \Big|_{p_i=0} p_i. \quad (2)$$

Formally, we may define L_i as the *linearity projection operator* with respect to p_i . Let us denote a generic polynomial of p_1, \dots, p_n as Q , which may have an index to indicate more than one polynomial. The operator L_i can be algebraically represented as $L_i[Q] = p_i E_i \cdot D_i[Q]$, where E_i is elimination of p_i by substituting zero and D_i is differentiation with respect to p_i . In addition, a and b mean some coefficients independent of p_i . We then have the following relations for these operators:

- $L_i[ap_i^n] = ap_i \delta_{1,n}$
- $L_i \cdot L_i[Q] = L_i[Q]$
- $L_i \cdot L_j[Q] = L_j \cdot L_i[Q]$
- $L_i[aQ_1 + bQ_2] = aL_i[Q_1] + bL_i[Q_2]$
- $E_j \cdot L_i[Q] = L_i \cdot E_j[Q]$
- $L_i[Z_j] = 0, \quad \text{if } i \leq j$

where $\delta_{1,n}$ is the Kronecker delta. It immediately follows that

$$Z_k = \left\{ \prod_{i=1}^k (1 - L_i) \right\} [Z_0]. \quad (3)$$

Let us now calculate the probability for the n th player to win the LUPI game by choosing number i . Then within the other $n - 1$ people, there should be no winner up to $i - 1$ and no one should choose the number i . The last condition can be easily imposed by the same trick. That is, we simply substitute zero for p_i to exclude every case where p_i appears. Therefore, the probability of our concern is simply written as $c_i(\vec{p}) = E_i[Z_{i-1}] = \left\{ \prod_{j=1}^{i-1} (1 - L_j) \right\} \cdot E_i[Z_0]$ and the n th player's expected payoff is

$$W(\vec{\pi}; \vec{p}) = \sum_{i=1}^n c_i(\vec{p}) \pi_i,$$

where $\vec{\pi} = (\pi_1, \pi_2, \dots, \pi_n)$ is the n th player's strategy.

For example,

$$c_1 = Z_0|_{p_1=0} = \left(\sum_{i \neq 1}^n p_i \right)^{n-1} = (1 - p_1)^{n-1}, \quad (4)$$

$$\begin{aligned} c_2 &= Z_1|_{p_2=0} = \left(\sum_{i \neq 2}^n p_i \right)^{n-1} - (n-1)p_1 \left(\sum_{i \neq 1,2}^n p_i \right)^{n-2} \\ &= (1 - p_2)^{n-1} - (n-1)p_1(1 - p_1 - p_2)^{n-2}, \end{aligned} \quad (5)$$

$$\begin{aligned} c_3 &= Z_2|_{p_3=0} \\ &= \left(\sum_{i \neq 3}^n p_i \right)^{n-1} - (n-1)p_1 \left(\sum_{i \neq 1,3}^n p_i \right)^{n-2} \\ &\quad - (n-1)p_1 \left(\sum_{i \neq 2,3}^n p_i \right)^{n-2} + (n-1)(n-2)p_1p_2 \left(\sum_{i \neq 1,2,3}^n p_i \right)^{n-3} \\ &= (1 - p_3)^{n-1} - (n-1)p_1(1 - p_1 - p_3)^{n-2} \\ &\quad - (n-1)p_2(1 - p_2 - p_3)^{n-2} \\ &\quad + (n-1)(n-2)p_1p_2(1 - p_1 - p_2 - p_3)^{n-3}. \end{aligned} \quad (6)$$

From the normalization condition, $p_n = 1 - \sum_{i=1}^{n-1} p_i$, indeed we have only $n - 1$ degrees of freedom in choosing \vec{p} so that

$$W(\vec{\pi}; \vec{p}) = \sum_{i=1}^{n-1} (c_i - c_n) \pi_i + c_n.$$

The NE solution, \vec{p}_{NE} , should satisfy $c_i(\vec{p}_{\text{NE}}) = \text{constant}$ for every i so that W cannot be better by changing $\vec{\pi}$. That is, the chance of winning is the same on all numbers. For $n = 3$, all the above calculation coincides with that presented in Ref. [4]. Note that we have $n - 1$ degrees of freedom and $n - 1$ equations so we can solve these simultaneously. Figure 1(a) shows solutions for some n values, obtained by using the Newton method. In Fig. 1(b), we plot its scaled version, considering that the horizontal axis naturally scales with n , and the vertical axis roughly with $1/n$.

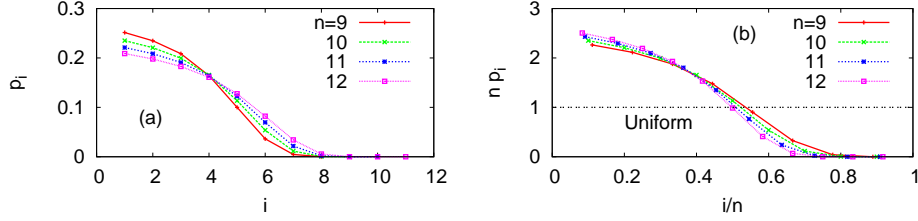


Fig 1. (a) NE strategies, \vec{p}_{NE} , with varying the number of players, n . (b) Scaled plots with respect to n . As n increase, the solution deviates more from the uniform solution (dotted).

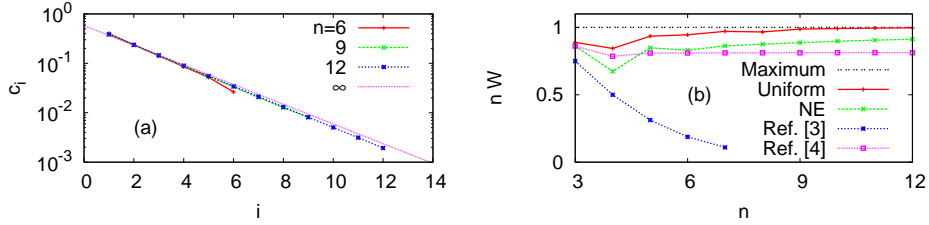


Fig 2. (a) The expected chance of winning by choosing i , when the other $n - 1$ players follow the uniform solution defined as $p_i = 1/n$. (b) Comparison of outcomes scaled with n . The uniform solution gives an outcome that rapidly converges to the theoretical maximum $W = 1/n$ (dotted), while the NE outcome remains below that. The other two curves represent outcomes from the strategies suggested in previous works.

Setting $c_1 = c_2$ from Eqs. (4) and (5), we find

$$(1 - p_2)^{n-1} - (1 - p_1)^{n-1} = (n - 1)p_1(1 - p_1 - p_2)^{n-2}. \quad (7)$$

Since the right-hand side should be positive, we can conclude that $p_2 < p_1$ for every finite n , which means the equilibrium solution cannot be uniform. Its implication is remarkable: Let us imagine that everyone employs the same strategy \vec{p} . In other words, $\vec{\pi} = \vec{p}$. The question is what the strategy \vec{p} should be in order to maximize the expected payoff, $W(\vec{p}; \vec{p})$. For example, for $n = 3$, we find

$$W(\vec{p}; \vec{p}) = (1 - p_1)(1 - p_2)(1 - p_3),$$

which is maximized at a uniform solution, $\vec{p}_* = (1/3, 1/3, 1/3)$. This is clearly different from the NE solution, $\vec{p}_{\text{NE}} = (2\sqrt{3} - 3, 2 - \sqrt{3}, 2 - \sqrt{3})$ [4]. The expression is not so simple for $n > 3$, but solving $\nabla_{\vec{p}} W(\vec{p}; \vec{p}) = 0$ under $\sum_i p_i = 1$, we still see that the uniform solution gives the best outcome to the population on average. Although we confirmed this only for $n \leq 12$, this result is plausible since the risk of overlapping choices will be minimized by the uniform solution. On the other hand, we already know that the equilibrium strategy cannot be uniform. Indeed, if all the people follow $\vec{p}_* = (1/n, \dots, 1/n)$, a player's chance of winning rapidly decays with her chosen number i [Fig. 2(a)]. For example, one may use the formulation above to obtain $c_i = e^{-1}(1 - e^{-1})^{i-1}$ for large n , which coincide with the multinomial analysis in Ref. [3]. Therefore, each player is motivated to deviate from this uniform

solution by choosing low numbers more frequently. In short, there exists a dilemma between the *best* and the *equilibrium* strategy. Note that since each play only has one winner, the theoretical upper limit for the gain per play is $1/n$. Let us then compare these three cases: the equilibrium outcome, the best possible outcome, and the theoretical upper limit [Fig 2(b)]. While the uniform solution quickly converges to the theoretical maximum, the NE solution remains suboptimal in this plot. For comparison, we also plot outcomes from the strategies in previous works: Ref. [3] suggested a strategy with $p_1 = p_2 = 1/2$ which gives $W(\vec{p}, \vec{p}) = 2^{1-n}$, while in Ref. [4] the solution is approximately written as $p_i = 2^{-i}$ for $i < n$ with $p_n = 2^{1-n}$. Note that the latter one actually performs better than presented in Ref. [4], since we have taken all the possibilities into consideration. In any case, these become smaller than the outcome from our NE solution.

As Eq. (3) indicates, the number of terms to consider increases exponentially as n grows, making the exact enumeration of probabilities intractable. Although our current computational resources allow us to arrive only up to $n = 12$, our procedure provides a systematic way to take all the combinatorial cases into account for any n , in principle. We believe that the behaviors shown in Fig. 2 give a clue to anticipate \vec{p}_{NE} in the limit of large n . Specifically, we sketch a way to perform this task: The NE solution is obtained when $c_i = c_j$ for all i and j . This means that we can set c_i to a certain constant c_0 for $i = 1, \dots, n$. The point is that c_i has only one more variable, p_i , than c_{i-1} , so that we can attack each variable p_i one by one. For example, Eq. (4) with $c_1 = c_0$ gives $p_1(c_0) = 1 - c_0^{1/(n-1)}$. Substituting this value into Eq. (5) and with $c_2 = c_0$ gives a value for $p_2(c_0)$, and so on. If the correct NE-value is chosen so that $c_0 = c_{\text{NE}}$ then the equilibrium solution, \vec{p}_{NE} is obtained. When there exist several real solutions for p_i then the correct root can be found using the restriction $0 < p_i < 1$ and the normalization condition $\sum_{j=1}^i p_j < 1$. The procedure for a three-players ($n = 3$) game, where we have $c_0 = c_{\text{NE}} = 28 - 16\sqrt{3} \approx 0.287$ (Fig. 2b), goes as follows:

$$\begin{aligned}
 c_1 &= c_0 = (1 - p_1)^2 \Rightarrow p_1 \approx 0.4643. \\
 c_2 &= c_0 = (1 - p_2)^2 - 2p_1(1 - p_1 - p_2) \\
 &\Rightarrow [\text{inserting } p_1] \Rightarrow p_2 \approx 0.2684. \\
 c_3 &= c_0 = (1 - p_3)^2 - 2p_1(1 - p_1 - p_3) - 2p_2(1 - p_2 - p_3) + 2p_1p_2 \\
 &\Rightarrow [\text{inserting } p_1 \text{ and } p_2] \Rightarrow p_3 \approx 0.2673 \pm 0.0246i.
 \end{aligned}$$

The fact that the last solution is complex indicates that the value of c_0 is not exactly correct. Adding more digits decreases the imaginary part while the real part moves closer to the correct solution (e.g., $c_0 = 0.287187$ gives $p_3 = 0.267949 - 5.04978 \times 10^{-4}i$). Since there is no general formula for roots of polynomials of degree larger than four, it seems that this procedure should be carried out numerically. Figure 3 shows the behavior of c_i for $i = 1$ to 4 and $n = 9$. The solution of p_i is obtained in each step by finding the crossing point $c_i = c_0$ (where p_{i-1} has been inserted into c_i).

To sum up, the benefit of this procedure is that if we know the correct NE-value, $c_0 = c_{\text{NE}}$, then we can calculate the probabilities one by one up to any i

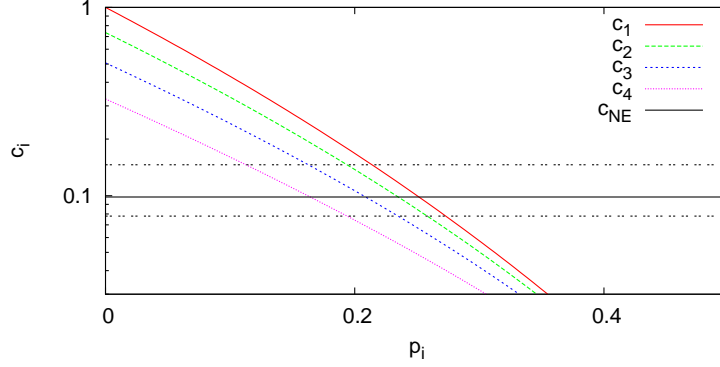


Fig 3. The expressions c_i as functions of p_i for $n = 9$ players. A Nash equilibrium solution is obtained when $c_i = c_{NE}$ for all $i \leq n$. p_1 is found by the condition $c_1 = c_{NE}$. Inserting this value into c_2 allows for a determination of p_2 as the crossing point between c_2 and c_{NE} , and so on. Or the value of c_{NE} can be obtained in this way by finding c_0 which fulfills the normalization condition $\sum_{i=1}^n p_i = 1$. The two dotted horizontal lines represent the interval within which c_{NE} must lie if we use the expressions up to c_4 instead of c_n .

we want or are capable of doing within our computational resources, for any n . In many situations it might, for example, be enough to know the solution up to certain i above which p_i 's are negligible. The method of solving for all i simultaneously instead results in an *all-or-nothing* situation. Of course, the down side is that we need to know c_{NE} in advance. However, recalling that $c_{NE} = W_{NE} \equiv W(\vec{p}_{NE}; \vec{p}_{NE})$, we see the limiting behavior $\lim_{n \rightarrow \infty} n c_{NE} = \lim_{n \rightarrow \infty} n W_{NE}$ in Fig. 2(b). Putting such a $c_0 = c_{NE}$ into this procedure may thus yield \vec{p}_{NE} for large n . The solution obtained in this way up to $i = 4$ for $n = 9$ and $c_0 = 0.0985 \approx c_{NE}$ (see Fig. 3) is $\vec{p} = (0.2515, 0.2349, 0.2087, 0.1643, \dots)$. This can be compared to the result obtained from solving for all i simultaneously, which is $\vec{p} = (0.2515, 0.2348, 0.2086, 0.1641, \dots)$.

This method could also be used to find the actual NE solution, without knowing c_{NE} , by iterating the procedure until a self-consistent solution is found for all i . If this is again beyond our computational power, one could at least determine an interval in which $c_0 = c_{NE}$ lies. Let's say that we use expressions for c_i up to $i = j$. If c_0 is too small then the normalization gets violated by $\sum_{i=1}^j p_i > 1$. If c_0 is too large then the crossing points result in probabilities that are too small to add up to one in total, as long as p_i is a monotonically decreasing function of i . Thus, the largest possible sum of all n probabilities is obtained if p_i is uniform for $i \geq j$, so the sum $\sum_{i=1}^j p_i + (n - j)p_j$ must be larger or equal to one. The interval obtained for $n = 9$ and $j = 4$ is marked by the two horizontal, dotted lines in Fig. 3 ($c_{0,min} = 0.078$ and $c_{0,max} = 0.146$). This interval becomes narrower as higher values of i are addressed. Note that when c_0 is changed all the curves c_i for $i > 1$ also change since we get new p_j 's with $j < i$ to be inserted into c_i .

3. Discussion

We have shown a way of taking all the combinatorial possibilities into account for the LUPI (lowest unique positive integer) game and proposed a procedure to find a

Nash equilibrium for a general number of players, n . It turns out, however, that the number of terms to consider grows exponentially in n , and that the computational capability of our personal computer can only solve the problem up to $n = 12$ players. To deal with this problem we also suggest an alternative way of solving for the equilibrium strategy for each value of i at a time, by inserting an estimate of the average expected payoff of each player. This procedure allows the problem to be solved sequentially instead of solving the whole problem simultaneously.

We also have found that a uniform solution distributes nearly optimal outcomes equally among all the players. However, this would be driven to a suboptimal equilibrium solution, so we may regard this n -person game as an example of a social dilemma to manifest a conflict between group and individual interests.

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